

極座標系における 2 階微分方程式

2次元極座標系のラプラシアン

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array}$$

$$\frac{\partial f}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial f}{\partial \theta}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial r}{\partial x} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial f}{\partial \theta} \right) \\ &= \frac{\partial^2 r}{\partial x^2} \frac{\partial f}{\partial r} + \frac{\partial r}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial r} \right) + \frac{\partial^2 \theta}{\partial x^2} \frac{\partial f}{\partial \theta} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial \theta} \right) \\ &= \frac{\partial^2 r}{\partial x^2} \frac{\partial f}{\partial r} + \frac{\partial r}{\partial x} \frac{\partial r}{\partial x} \frac{\partial^2 f}{\partial r^2} + \frac{\partial r}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial^2 \theta}{\partial x^2} \frac{\partial f}{\partial \theta} + \frac{\partial \theta}{\partial x} \frac{\partial r}{\partial x} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial^2 f}{\partial \theta^2} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial^2 r}{\partial y^2} \frac{\partial f}{\partial r} + \frac{\partial r}{\partial y} \frac{\partial r}{\partial y} \frac{\partial^2 f}{\partial r^2} + \frac{\partial r}{\partial y} \frac{\partial \theta}{\partial y} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial^2 \theta}{\partial y^2} \frac{\partial f}{\partial \theta} + \frac{\partial \theta}{\partial y} \frac{\partial r}{\partial y} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial y} \frac{\partial^2 f}{\partial \theta^2} \end{aligned}$$

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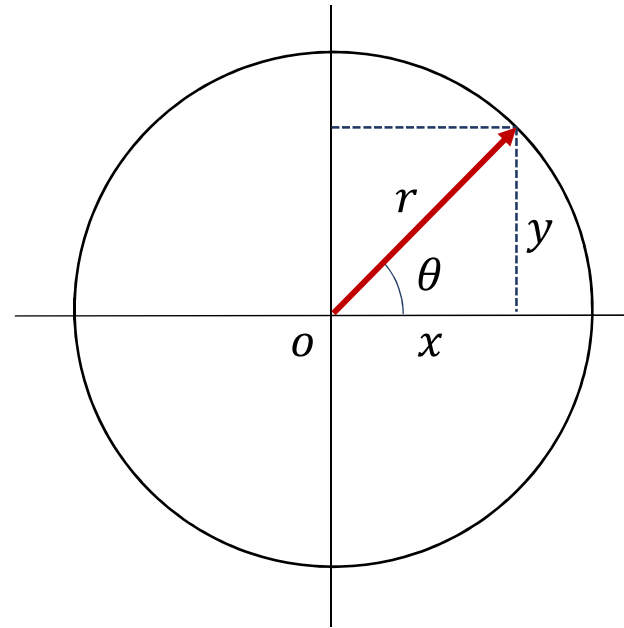
テキスト: p. 31

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x} \sqrt{x^2 + y^2} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y} \sqrt{x^2 + y^2} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x} \tan^{-1} \frac{y}{x} = \frac{-y}{r^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y} \tan^{-1} \frac{y}{x} = \frac{x}{r^2}$$



$$\frac{\partial}{\partial x} \tan \theta = \frac{\partial}{\partial x} \frac{y}{x} = -\frac{y}{x^2}$$

$$= \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} \tan \theta = \frac{\partial \theta}{\partial x} (1 + \tan^2 \theta)$$

$$\frac{\partial}{\partial y} \tan \theta = \frac{\partial}{\partial y} \frac{y}{x} = \frac{1}{x} = \frac{\partial \theta}{\partial y} (1 + \tan^2 \theta)$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{x} \frac{1}{1 + \frac{y^2}{x^2}} = \frac{x}{r^2}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \frac{1}{1 + \frac{y^2}{x^2}} = \frac{-y}{r^2}$$

極座標系における 2 階微分方程式

テキスト: p. 32

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial r}{\partial x} \right) = \frac{\partial}{\partial x} \frac{x}{r} = \frac{1}{r} - \frac{x}{r^2} \frac{\partial r}{\partial x} = \frac{1}{r} - \frac{x}{r^2} \frac{x}{r} = \frac{y^2}{r^3}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial r}{\partial y} \right) = \frac{\partial}{\partial y} \frac{y}{r} = \frac{1}{r} - \frac{y}{r^2} \frac{\partial r}{\partial y} = \frac{1}{r} - \frac{y}{r^2} \frac{y}{r} = \frac{x^2}{r^3}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \theta}{\partial x} \right) = \frac{\partial}{\partial x} \frac{-y}{r^2} = -2 \frac{-y}{r^3} \frac{x}{r} = \frac{2xy}{r^4}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial \theta}{\partial y} \right) = \frac{\partial}{\partial y} \frac{x}{r^2} = -2 \frac{x}{r^3} \frac{y}{r} = \frac{-2xy}{r^4}$$

極座標系における 2 階微分方程式

テキスト: p. 31

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} & \frac{\partial \theta}{\partial x} &= \frac{-y}{r^2} & \frac{\partial^2 r}{\partial x^2} &= \frac{y^2}{r^3} & \frac{\partial^2 \theta}{\partial x^2} &= \frac{2xy}{r^4} \\ \frac{\partial r}{\partial y} &= \frac{y}{r} & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} & \frac{\partial^2 r}{\partial y^2} &= \frac{x^2}{r^3} & \frac{\partial^2 \theta}{\partial y^2} &= \frac{-2xy}{r^4} \end{aligned}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 r}{\partial x^2} \frac{\partial f}{\partial r} + \frac{\partial r}{\partial x} \frac{\partial r}{\partial x} \frac{\partial^2 f}{\partial r^2} + \frac{\partial r}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial^2 \theta}{\partial x^2} \frac{\partial f}{\partial \theta} + \frac{\partial \theta}{\partial x} \frac{\partial r}{\partial x} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 r}{\partial y^2} \frac{\partial f}{\partial r} + \frac{\partial r}{\partial y} \frac{\partial r}{\partial y} \frac{\partial^2 f}{\partial r^2} + \frac{\partial r}{\partial y} \frac{\partial \theta}{\partial y} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial^2 \theta}{\partial y^2} \frac{\partial f}{\partial \theta} + \frac{\partial \theta}{\partial y} \frac{\partial r}{\partial y} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial y} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2}{r^3} \frac{\partial f}{\partial r} + \frac{x^2}{r^2} \frac{\partial^2 f}{\partial r^2} + 2 \frac{x-y}{r r^2} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{2xy}{r^4} \frac{\partial f}{\partial \theta} + \frac{y^2}{r^4} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2}{r^3} \frac{\partial f}{\partial r} + \frac{y^2}{r^2} \frac{\partial^2 f}{\partial r^2} + 2 \frac{y x}{r r^2} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{-2xy}{r^4} \frac{\partial f}{\partial \theta} + \frac{x^2}{r^4} \frac{\partial^2 f}{\partial \theta^2}$$

極座標系における 2 階微分方程式

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\begin{aligned}
 &= \frac{y^2}{r^3} \frac{\partial f}{\partial r} + \frac{x^2}{r^2} \frac{\partial^2 f}{\partial r^2} + 2 \frac{x-y}{r r^2} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{2xy}{r^4} \frac{\partial f}{\partial \theta} + \frac{y^2}{r^4} \frac{\partial^2 f}{\partial \theta^2} \\
 &\quad + \frac{x^2}{r^3} \frac{\partial f}{\partial r} + \frac{y^2}{r^2} \frac{\partial^2 f}{\partial r^2} + 2 \frac{y x}{r r^2} \frac{\partial^2 f}{\partial \theta \partial r} + \frac{-2xy}{r^4} \frac{\partial f}{\partial \theta} + \frac{x^2}{r^4} \frac{\partial^2 f}{\partial \theta^2} \\
 &= \frac{r^2}{r^3} \frac{\partial f}{\partial r} + \frac{r^2}{r^2} \frac{\partial^2 f}{\partial r^2} + \frac{r^2}{r^4} \frac{\partial^2 f}{\partial \theta^2} \\
 &= \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}
 \end{aligned}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

ベッセルの微分方程式

$$\nabla^2 f = -\lambda f$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} = -\lambda f$$

$$f = R(r)\Phi(\theta)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) R(r)\Phi(\theta) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} R(r)\Phi(\theta) = -\lambda R(r)\Phi(\theta)$$

$$\frac{r^2}{R(r)} \left(\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} \right) + \lambda r^2 = -\frac{1}{\Phi(\theta)} \frac{\partial^2 \Phi(\theta)}{\partial \theta^2} = \beta$$

$$\left(\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} \right) + \lambda = \frac{\beta}{r^2} R(r)$$

$$\left(\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} \right) + \left(\lambda - \frac{n^2}{r^2} \right) R(r) = 0$$

$$\frac{\partial^2 \Phi(\theta)}{\partial \theta^2} = -\beta \Phi(\theta)$$

$$\Phi(\theta) = \cos n\theta, \sin n\theta$$

$$\Phi(\theta + 2\pi) = \Phi(\theta)$$

$$\beta = n^2$$

ベッセルの微分方程式

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} + \left(\lambda - \frac{n^2}{r^2} \right) R(r) = 0$$

$$x = r\sqrt{\lambda}$$

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{1}{x} \frac{\partial y(x)}{\partial x} + \left(1 - \frac{n^2}{x^2} \right) y(x) = 0$$

$$x^2 \frac{\partial^2 y(x)}{\partial x^2} + x \frac{\partial y(x)}{\partial x} + (x^2 - n^2) y(x) = 0$$

$$y(x) = \sum_{m=0}^{\infty} c_m x^{m+r}$$

$$\begin{aligned} \sum_{m=0}^{\infty} (m+r)(m+r-1)c_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)c_m x^{m+r} \\ + \sum_{m=0}^{\infty} c_m x^{m+r+2} + n^2 \sum_{m=0}^{\infty} c_m x^{m+r} = 0 \end{aligned}$$

ベッセルの微分方程式

テキスト: p. 33

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)c_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)c_m x^{m+r} + \sum_{m=0}^{\infty} c_m x^{m+r+2} + n^2 \sum_{m=0}^{\infty} c_m x^{m+r} = 0$$

r 次

$$r(r-1)c_0 + rc_0 - n^2c_0 = 0$$

$$(r^2 - n^2)c_0 = (r-n)(r+n)c_0 = 0$$

$$r = \pm n$$

$r+1$ 次

$$r(r+1)c_1 + (r+1)c_1 - n^2c_1 = \{(r+1)^2 - n^2\}c_1 = 0$$

$$c_1 = 0$$

$r+2$ 次
以上

$$(r+m)(r+m-1)c_m + (r+m)c_m + c_{m-2} - n^2c_m$$

$$= \{(r+m)^2 - n^2\}c_m + c_{m-2} = \underline{m(2n+m)c_m + c_{m-2}} = 0$$

$$c_m = \frac{-1}{m(2n+m)} c_{m-2}$$

ベッセル関数

テキスト: p. 34

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{1}{x} \frac{\partial y(x)}{\partial x} + \left(1 - \frac{n^2}{x^2}\right) y(x) = 0$$

$$c_m = \frac{-1}{m(2n+m)} c_{m-2}$$

$$c_0 = \frac{1}{2^n n!} \quad c_{2l+1} = 0$$

$$c_{2l} = \frac{(-1)^l}{2^{2l+n} l! (n+l)!}$$

$$J_n(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+n} l! (n+l)!} x^{2l+n}$$

$$c_{2l+1} = 0$$

$$\begin{aligned} c_{2l} &= \frac{-1}{2^{2l}(n+l)} c_{2l-2} \\ &= \frac{(-1)^2}{2^{4l}(l-1)(n+l)(n+l-1)} c_{2l-4} \\ &= \frac{(-1)^l n!}{2^{2l} l! (n+l)!} c_0 \end{aligned}$$

$$J_0(x) = 1 - \frac{1}{2^2(1!)^2} x^2 + \frac{1}{2^4(2!)^2} x^4 - \frac{1}{2^6(3!)^2} x^6 + \frac{1}{2^8(4!)^2} x^8 - \dots$$

$$J_1(x) = \frac{1}{2} x - \frac{1}{2^3 1! 2!} x^3 + \frac{1}{2^5 2! 3!} x^5 - \frac{1}{2^7 3! 4!} x^7 + \frac{1}{2^9 4! 5!} x^9 - \dots$$

ベッセル関数

$$J_0(x) = 1 - \frac{1}{2^2(1!)^2} x^2 + \frac{1}{2^4(2!)^2} x^4 - \frac{1}{2^6(3!)^2} x^6 + \frac{1}{2^8(4!)^2} x^8 - \dots$$

$$J_1(x) = \frac{1}{2}x - \frac{1}{2^3 1! 2!} x^3 + \frac{1}{2^5 2! 3!} x^5 - \frac{1}{2^7 3! 4!} x^7 + \frac{1}{2^9 4! 5!} x^9 - \dots$$

