

ベッセル関数の積分表示

テキスト: p. 39

$$\exp\left\{\frac{x}{2}(t - t^{-1})\right\} = J_0(x) + \{J_1(x)t + J_{-1}(x)t^{-1}\} + \dots \\ + \{J_n(x)t^n + J_{-n}(x)t^{-n}\} + \dots$$

$$t = e^{i\theta} \quad \Rightarrow \quad t - t^{-1} = 2i \sin \theta$$

$$J_n(x) = (-1)^n J_{-n}(x) \quad \Rightarrow \quad J_n(x)t^n + J_{-n}(x)t^{-n} = J_n(x)(e^{i\theta} + (-1)^n e^{-i\theta})$$

$$\exp\{ix \sin \theta\} = \cos(x \sin \theta) + i \sin(x \sin \theta) \\ = J_0(x) + 2iJ_1(x) \sin \theta + 2J_2(x) \cos 2\theta \\ + 2iJ_3(x) \sin 3\theta + 2J_4(x) \cos 4\theta + \dots$$

ベッセル関数の積分表示

$$J_{2l}(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta) \cos\{2l\theta\} d\theta \\ J_{2l+1}(x) = \frac{1}{\pi} \int_0^\pi \sin(x \sin \theta) \sin\{(2l+1)\theta\} d\theta$$

フーリエ級数展開

テキスト:p.7

$$f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx' + \sum_{n=1}^{\infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx' \right] \sin nx \\ + \sum_{n=1}^{\infty} \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx' \right] \cos nx$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx'$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx'$$

ガンマ関数

テキスト: p. 40

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx \quad (s > 0)$$

$$\int_0^{\infty} e^{-x} x^{s-1} dx = \int_0^{\infty} \left\{ \frac{d}{dx} \underbrace{(-e^{-x} x^{s-1})}_{0 \text{ (} s > 1 \text{)}} \right\} dx + (s-1) \int_0^{\infty} e^{-x} x^{s-2} dx$$

$$\Gamma(1) = \int_0^{\infty} e^{-x} dx = 1$$

$$\Gamma(s) = (s-1)\Gamma(s-1) \quad (s > 1)$$

$$\Gamma(2) = (2-1)\Gamma(1) = 1$$

$$\Gamma(3) = (3-1)\Gamma(2) = 2!$$

$$\Gamma(4) = (4-1)\Gamma(3) = 3!$$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)!$$

$$\Gamma(1/2) = \int_0^{\infty} e^{-x} x^{-1/2} dx$$

$$= 2 \int_0^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$$

$$x = \xi^2 \quad dx = 2\xi d\xi$$

s が負の場合の定義

$$\Gamma(s-1) = (s-1)^{-1} \Gamma(s)$$

$$\Gamma(3/2) = (3/2-1)\Gamma(3/2-1) = \frac{\sqrt{\pi}}{2}$$

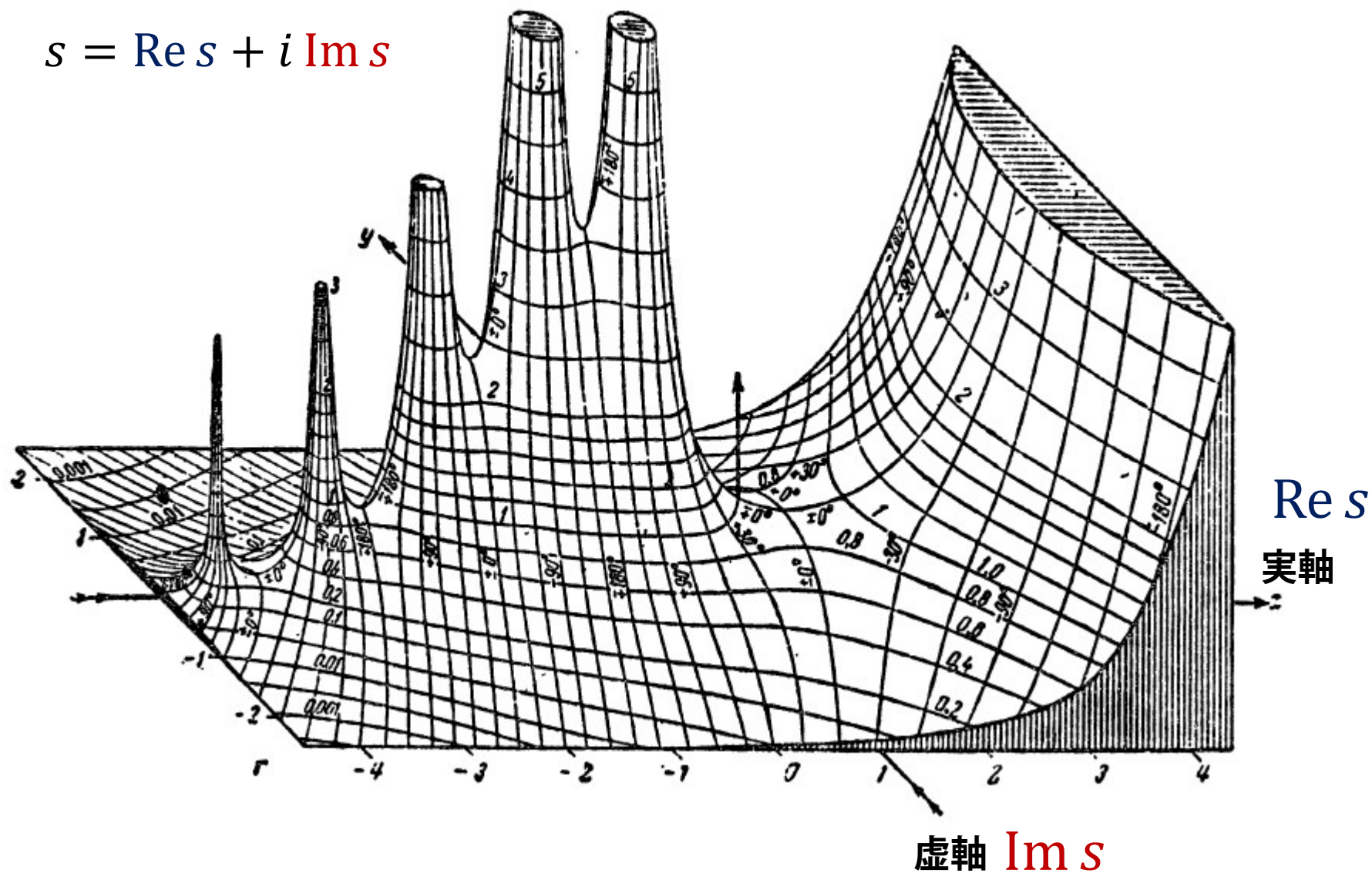
$$\Gamma(-3)^{-1} = \Gamma(-2)^{-1} = \Gamma(-1)^{-1} = \Gamma(0)^{-1} = 0$$

ガンマ関数

テキスト:p. 40

ガンマ関数の絶対値 $|\Gamma(s)|$

$$s = \text{Re } s + i \text{Im } s$$



一般のベッセル関数

テキスト: p. 41

整数 $n \rightarrow$ 実数 ν

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{1}{x} \frac{\partial y(x)}{\partial x} + \left(1 - \frac{\nu^2}{x^2}\right) y(x) = 0$$

$$y(x) = \sum_{m=0}^{\infty} c_m x^{m+r}$$

代入

$$x^2 \frac{\partial^2 y(x)}{\partial x^2} + x \frac{\partial y(x)}{\partial x} + (x^2 - \nu^2) y(x) = 0$$

$$\begin{aligned} \sum_{m=0}^{\infty} (m+r)(m+r-1) c_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) c_m x^{m+r} \\ + \sum_{m=0}^{\infty} c_m x^{m+r+2} + \nu^2 \sum_{m=0}^{\infty} c_m x^{m+r} = 0 \end{aligned}$$

ベッセルの微分方程式

テキスト:p.41

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)c_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)c_m x^{m+r} + \sum_{m=0}^{\infty} c_m x^{m+r+2} + v^2 \sum_{m=0}^{\infty} c_m x^{m+r} = 0$$

r 次

$$r(r-1)c_0 + rc_0 - v^2 c_0 = 0$$

$$(r^2 - v^2)c_0 = (r-v)(r+v)c_0 = 0$$

$$r = \pm v$$

$r+1$ 次

$$r(r+1)c_1 + (r+1)c_1 - v^2 c_1 = \{(r+1)^2 - v^2\}c_1 = 0$$

$$c_1 = 0$$

$r+2$ 次
以上

$$(r+m)(r+m-1)c_m + (r+m)c_m + c_{m-2} - v^2 c_m$$

$$= \{(r+m)^2 - v^2\}c_m + c_{m-2} = \underline{m(2v+m)c_m + c_{m-2} = 0}$$

$r = v$ の場合

$$c_m = \frac{-1}{m(2v+m)} c_{m-2}$$

一般のベッセル関数

テキスト: p. 41

$$\frac{\partial^2 y(x)}{\partial x^2} + \frac{1}{x} \frac{\partial y(x)}{\partial x} + \left(1 - \frac{\nu^2}{x^2}\right) y(x) = 0$$

$$\Gamma(s+1) = s \Gamma(s)$$

$r = \nu$ の場合

$$c_0 = \frac{1}{2^\nu \Gamma(\nu+1)} \quad c_{2l+1} = 0$$

$$c_{2l} = \frac{(-1)^l}{2^{2l+\nu} l! \Gamma(\nu+l+1)}$$

$$J_\nu(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+\nu} l! \Gamma(\nu+l+1)} x^{2l+\nu}$$

$$c_m = \frac{-1}{m(2\nu+m)} c_{m-2} \quad m = 2l$$

$$c_{2l} = \frac{-1}{2^{2l} l(\nu+l)} c_{2l-2}$$

$$= \frac{-\Gamma(\nu+l)}{2^{2l} l \Gamma(\nu+l+1)} c_{2l-2}$$

$$= \frac{(-1)^l \Gamma(\nu+1)}{2^{2l} l! \Gamma(\nu+l+1)} c_0$$

$r = -\nu$ の場合

$$J_{-\nu}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l-\nu} l! \Gamma(-\nu+l+1)} x^{2l-\nu}$$

$$c_m = \frac{-1}{m(-2\nu+m)} c_{m-2}$$

ベッセル関数の漸化式

テキスト: p. 42

$$x^{-\nu} J_{\nu}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+\nu} l! \Gamma(\nu + l + 1)} x^{2l}$$

$$\frac{d}{dx} (x^{-\nu} J_{\nu}(x)) = \sum_{l=1}^{\infty} \frac{2l(-1)^l}{2^{2l+\nu} l! \Gamma(\nu + l + 1)} x^{2l-1}$$

$$= -\frac{x}{2} \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{2^{2(l-1)+\nu} (l-1)! \Gamma(\nu + l + 1)} x^{2(l-1)}$$

$$= -\frac{x}{2} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+\nu} l! \Gamma(\nu + l + 2)} x^{2l}$$

$$= -x \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+(\nu+1)} l! \Gamma((\nu + 1) + l + 1)} x^{2l}$$

$$= -x (x^{-(\nu+1)}) \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+(\nu+1)} l! \Gamma((\nu + 1) + l + 1)} x^{2l+(\nu+1)}$$

$$= -x^{-\nu} J_{\nu+1}(x)$$

$$\frac{d}{dx} (x^{-\nu} J_{\nu}(x)) = -x^{-\nu} J_{\nu+1}(x)$$

ベッセル関数の漸化式

テキスト: p. 42

$$x^\nu J_\nu(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+\nu} l! \Gamma(\nu + l + 1)} x^{2l+2\nu}$$

$$\frac{d}{dx} (x^\nu J_\nu(x)) = \sum_{l=0}^{\infty} \frac{2(l+\nu)(-1)^l}{2^{2l+\nu} l! \Gamma(\nu + l + 1)} x^{2l+2\nu-1}$$

$$= \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+\nu-1} l! \Gamma(\nu + l)} x^{2l+2\nu-1}$$

$$= x^\nu \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+\nu-1} l! \Gamma((\nu-1) + l + 1)} x^{2l+\nu-1}$$

$$= x^\nu J_{\nu-1}(x)$$

$$\frac{d}{dx} (x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$$

$$\frac{d}{dx} (x^{-\nu} J_\nu(x)) = -\frac{\nu}{x} x^{-\nu} J_\nu(x) + x^{-\nu} \frac{d}{dx} J_\nu(x) = -x^{-\nu} J_{\nu+1}(x)$$

$$\frac{d}{dx} (x^\nu J_\nu(x)) = \frac{\nu}{x} x^\nu J_\nu(x) + x^\nu \frac{d}{dx} J_\nu(x) = x^\nu J_{\nu+1}(x)$$

ベッセル関数の漸化式

テキスト: p. 43

$$\frac{d}{dx} (x^{-\nu} J_{\nu}(x)) = -x^{-\nu} J_{\nu+1}(x)$$

$$\frac{d}{dx} (x^{\nu} J_{\nu}(x)) = x^{\nu} J_{\nu-1}(x)$$

$$\frac{d}{dx} (x^{-\nu} J_{\nu}(x)) = -\frac{\nu}{x} x^{-\nu} J_{\nu}(x) + x^{-\nu} \frac{d}{dx} J_{\nu}(x) = -x^{-\nu} J_{\nu+1}(x)$$

$$\frac{d}{dx} (x^{\nu} J_{\nu}(x)) = \frac{\nu}{x} x^{\nu} J_{\nu}(x) + x^{\nu} \frac{d}{dx} J_{\nu}(x) = x^{\nu} J_{\nu-1}(x)$$

漸化式

$$-\frac{\nu}{x} J_{\nu}(x) + \frac{d}{dx} J_{\nu}(x) = -J_{\nu+1}(x)$$

$$\frac{\nu}{x} J_{\nu}(x) + \frac{d}{dx} J_{\nu}(x) = J_{\nu-1}(x)$$



$$2 \frac{d}{dx} J_{\nu}(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$$

$$\frac{2\nu}{x} J_{\nu}(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$$

第2種、第3種ベッセル関数

2階ベッセルの微分方程式の一般解 $C_1 J_\nu(x) + C_2 J_{-\nu}(x)$

ν が整数 n の場合 $J_{-n}(x) = (-1)^n J_n(x)$ より、別の解が存在

第2種ベッセル関数 $Y_n(x)$ ・ノイマン関数 $N_n(x)$

$$N_n(x) = Y_n(x) = \lim_{\nu \rightarrow n} \frac{1}{\sin \nu \pi} \{ \cos \nu \pi J_\nu(x) - J_{-\nu}(x) \}$$

第3種ベッセル関数 ・ハンケル関数

$$N_n^{(1)}(x) = J_n(x) + iY_n(x)$$

$$N_n^{(2)}(x) = J_n(x) - iY_n(x)$$

ロピタルの定理と

$$\frac{d}{d\nu} x^\nu = x^\nu \ln x$$

より、 $\ln x$ が現れる

ベッセル関数の漸近形

$$\begin{aligned}
 y(x) &= \sqrt{x} J_n(x) & \frac{d^2 y(x)}{dx^2} &= \frac{d}{dx} \left(\frac{1}{2} \frac{1}{\sqrt{x}} J_n(x) + \sqrt{x} \frac{dJ_n(x)}{dx} \right) \\
 & & &= -\frac{1}{4} \frac{1}{x\sqrt{x}} J_n(x) + \frac{1}{\sqrt{x}} \frac{dJ_n(x)}{dx} + \sqrt{x} \frac{d^2 J_n(x)}{dx^2}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{\sqrt{x}} \frac{dJ_n(x)}{dx} + \sqrt{x} \frac{d^2 J_n(x)}{dx^2} &= \sqrt{x} \left(\frac{d^2 J_n(x)}{dx^2} + \frac{1}{x} \frac{dJ_n(x)}{dx} \right) \\
 &= -\sqrt{x} \left(1 - \frac{n^2}{x^2} \right) J_n(x) = -\left(1 - \frac{n^2}{x^2} \right) y_n(x)
 \end{aligned}$$

$$\begin{aligned}
 \frac{d^2 y(x)}{dx^2} &= -\frac{1}{4} \frac{1}{x^2} y(x) + \left(1 - \frac{n^2}{x^2} \right) y_n(x) & \frac{d^2 y(x)}{dx^2} &\cong -y_n(x) \quad (x \gg 1) \\
 &= -\left(1 - \frac{4n^2 - 1}{4x^2} \right) y_n(x) & y_n(x) &= \sqrt{x} J_n(x) = \cos(x + \alpha) \\
 & & J_n(x) &= \frac{1}{\sqrt{x}} \cos(x + \alpha)
 \end{aligned}$$

ベッセル関数の漸近形

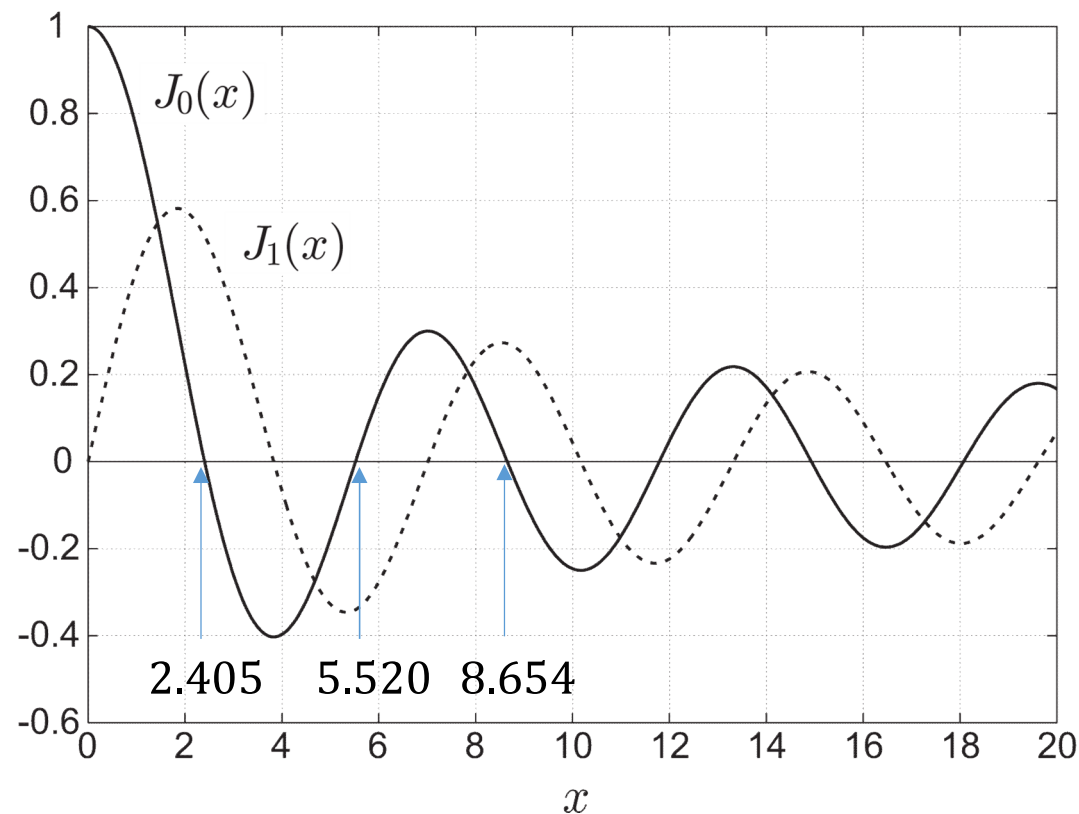
テキスト: p. 44

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$N_n^{(1)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \exp\left\{i\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right)\right\}$$

$$Y_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$N_n^{(2)}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \exp\left\{-i\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right)\right\}$$



3次元極座標系における2階微分方程式

3次元極座標系のラプラシアン

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2}$$

$$\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \cancel{\frac{\partial^2}{\partial \rho^2}} = \cancel{\frac{\partial^2 f}{\partial \rho^2}} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r \sin \theta} \frac{\partial f}{\partial \rho} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

$$x = r \sin \theta \cos \phi$$

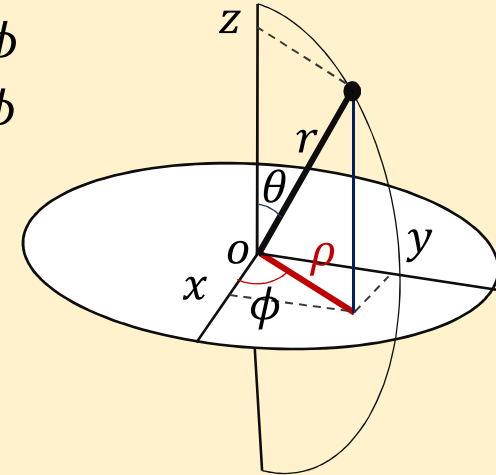
$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\rho = r \sin \theta$$

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$



$$z = r \cos \theta$$

$$\rho = r \sin \theta$$

3次元極座標系のラプラシアン

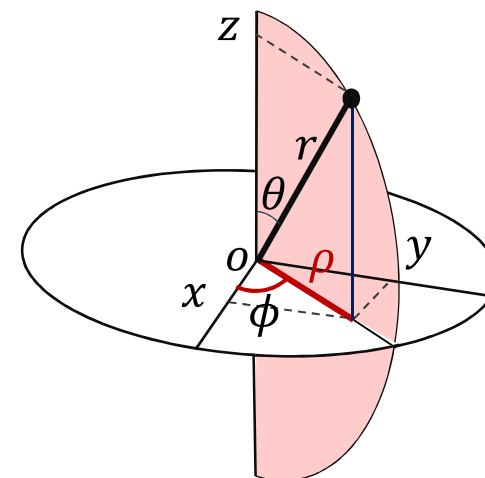
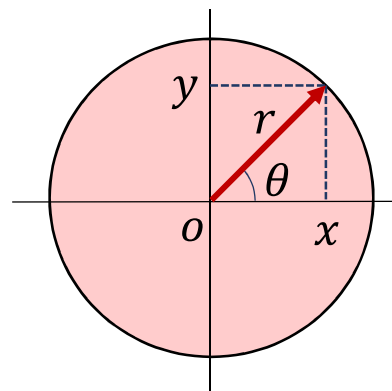
テキスト:p.46

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{r \sin \theta} \frac{\partial f}{\partial \rho} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}$$

$$= \frac{1}{r \sin \theta} \left(\sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta} \right) + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = r \cos \theta$$

$$\rho = r \sin \theta$$

$$\frac{\partial f}{\partial \rho} = \frac{\partial r}{\partial \rho} \frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial \rho} \frac{\partial f}{\partial \theta} = \sin \theta \frac{\partial f}{\partial r} + \frac{\cos \theta}{r} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y} \sqrt{x^2 + y^2} = \frac{y}{r}$$

$$\frac{\partial r}{\partial \rho} = \frac{\rho}{r} = \sin \theta$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y} \tan^{-1} \frac{y}{x} = \frac{x}{r^2}$$

$$\frac{\partial \theta}{\partial \rho} = \frac{z}{r^2} = \frac{\cos \theta}{r}$$