

ルジャンドルの微分方程式

$$\nabla^2 f = -\lambda f$$

$$\nabla^2 f = Y(\theta, \phi) \frac{\partial^2 R(r)}{\partial r^2} + Y(\theta, \phi) \frac{2}{r} \frac{\partial R(r)}{\partial r} + R(r) \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y(\theta, \phi)}{\partial \theta} \right) + R(r) \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 Y(\theta, \phi)}{\partial \phi^2} = -\lambda R(r) Y(\theta, \phi)$$

$$f = R(r) Y(\theta, \phi)$$

$$\frac{r^2}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{2r}{R(r)} \frac{\partial R(r)}{\partial r} + \lambda r^2 = \text{分離定数 } \mu = -\frac{1}{Y(\theta, \phi)} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y(\theta, \phi)}{\partial \theta} \right) - \frac{1}{Y(\theta, \phi)} \frac{1}{\sin^2 \theta} \frac{\partial^2 Y(\theta, \phi)}{\partial \phi^2}$$

$$\lambda = 0 \longrightarrow \frac{\partial^2 R(r)}{\partial r^2} + \frac{2}{r} \frac{\partial R(r)}{\partial r} - \frac{\mu}{r^2} R(r) = 0 \qquad t(t-1) + 2t - \mu = 0$$

$$R(r) = r^t \qquad t(t-1)r^{t-2} + 2tr^{t-2} - \mu r^{t-2} = 0 \qquad t(t+1) - \mu = 0$$

$$\text{一般解} \quad R(r) = Ar^n + B \frac{1}{r^{n+1}} \quad \longleftarrow \quad t = n, -(n+1) \quad \longleftarrow \quad \mu = n(n+1)$$

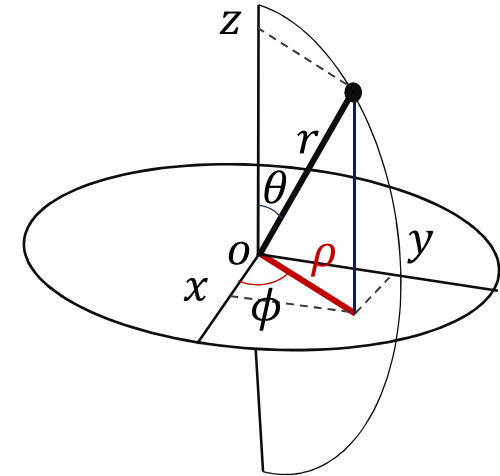
ルジャンドルの微分方程式

$Y(\theta, \phi)$ に ϕ 依存性がない場合

$$-\frac{1}{Y(\theta, \phi)} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y(\theta, \phi)}{\partial \theta} \right) = \mu = -n(n+1)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dY(\theta)}{d\theta} \right) + n(n+1)Y(\theta, \phi) = 0$$

$$x = \cos \theta \quad dx = -\sin \theta d\theta = -\sqrt{1-x^2} d\theta$$



$$\frac{d}{dx} \left((1-x^2) \frac{dY(x)}{dx} \right) + n(n+1)Y(x) = 0$$

Legendre の微分方程式

ルジャンドル関数

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

$$(1 - x^2) \frac{dP(x)}{dx} =$$

$$a_1 + 2a_2x + (3a_3 - a_1)x^2 + (4a_4 - 2a_2)x^3 + \cdots + \{(k+1)a_{k+1} - (k-1)a_{k-1}\}x^k + \cdots$$

$$\frac{d}{dx} \left((1 - x^2) \frac{dY(x)}{dx} \right) + n(n+1)Y(x) = 0$$

$$2a_2 + n(n+1)a_0 + \{2(3a_3 - a_1) + n(n+1)a_1\}x + \{3(4a_4 - 2a_2) + n(n+1)a_2\}x^2 + \\ \cdots + \{(k+2)(k+1)a_{k+2} - (k+1)ka_k + n(n+1)a_k\}x^k + \cdots = 0$$

$$2 \cdot 1a_2 - \{1 \cdot 0 - n(n+1)\}a_0 = 0$$

$$3 \cdot 2a_3 - \{2 \cdot 1 - n(n+1)\}a_1 = 0$$

$$4 \cdot 3a_4 - \{3 \cdot 2 - n(n+1)\}a_2 = 0$$

$$5 \cdot 6a_5 - \{4 \cdot 3 - n(n+1)\}a_3 = 0$$

$$(k+2)(k+1)a_{k+2} - \{(k+1)k - n(n+1)\}a_k = 0$$

ルジャンドル関数

$$(k+2)(k+1)a_{k+2} - \{(k+1)k - n(n+1)\}a_k = 0$$

 n が偶数の場合

$$2 \cdot 1a_2 - \{1 \cdot 0 - n(n+1)\}a_0 = 0$$

$$4 \cdot 3a_4 - \{3 \cdot 2 - n(n+1)\}a_2 = 0$$

$$P_n(x) = a_0 + a_2x^2 + \cdots + a_nx^n$$

 n が奇数の場合

$$3 \cdot 2a_3 - \{2 \cdot 1 - n(n+1)\}a_1 = 0$$

$$5 \cdot 6a_5 - \{4 \cdot 3 - n(n+1)\}a_3 = 0$$

$$P_n(x) = a_1x^1 + a_3x^3 + \cdots + a_nx^n$$

$$a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{(2n-1)!!}{n!} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$$

$$k(k-1)a_k - \{(k-1)(k-2) - n(n+1)\}a_{k-2} = 0$$



$$a_{k-2} = -\frac{k(k-1)}{(n-k+2)(n+k-1)}a_k \quad \Rightarrow \quad a_{n-2} = -\frac{n(n-1)}{2(2n-1)}a_n$$

ルジャンドル関数

テキスト:p. 48

$$P_n(x) = a_0 + a_2x^2 + \cdots + a_nx^n$$

$$a_n = \frac{(2n-1)!!}{n!}$$

$$\begin{aligned} a_{n-2} &= -\frac{n(n-1)}{2(2n-1)} a_n \\ &= -\frac{n(n-1)}{2(2n-1)} \frac{(2n-1)!!}{n!} \\ &= -\frac{(2n-3)!!}{2 \cdot 1! (n-2)!} \end{aligned}$$

ルジャンドル関数

$$P_n(x) = \sum_{m=0}^{[n/2]} (-1)^m \frac{(2n-2m-1)!!}{2^m \cdot m! (n-2m)!} x^{n-2m}$$

$$a_{k-2} = -\frac{k(k-1)}{(n-k+2)(n+k-1)} a_k$$

$$\begin{aligned} a_{n-4} &= -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2} \\ &= \frac{(n-2)(n-3)}{4(2n-3)} \frac{(2n-3)!!}{2 \cdot 1! (n-2)!} \\ &= \frac{(2n-5)!!}{2^2 \cdot 2! (n-4)!} \end{aligned}$$

$$a_{n-2m} = (-1)^m \frac{(2n-2m-1)!!}{2^m \cdot m! (n-2m)!}$$

$[n/2]$ は $n/2$ を超えない整数

ルジャンドル関数

$P_0(x) = 1$ テキスト: p. 49

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(2n-2m-1)!!}{2^m \cdot m! (n-2m)!} x^{n-2m}$$

$$P_n(-x) = (-1)^n P_n(x)$$

