

ルジャンドル関数の母関数

テキスト: p. 50

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n = P_0(x) + P_1(x)t + P_2(x)t^2 + \cdots + P_n(x)t^n + \cdots$$

$$\frac{1}{\sqrt{1-X}} = 1 + \frac{1}{2}X + \frac{1}{2} \frac{3}{2} \frac{1}{2!} X^2 + \frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{1}{3!} X^3 + \cdots + \frac{(2k-1)!!}{2^k k!} X^k + \cdots$$

$$\begin{aligned} X^k &= (2tx - t^2)^k = \sum_{l=0}^k \frac{(-1)^{k-l} k!}{l! (k-l)!} (2tx)^l t^{2(k-l)} \\ &= \sum_{l=0}^k \frac{(-1)^{k-l} k!}{l! (k-l)!} 2^l x^l t^{2k-l} \end{aligned}$$

$$\begin{aligned} \frac{1}{\sqrt{1-2tx+t^2}} &= \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{(2k-1)!!}{2^k k!} \frac{(-1)^{k-l} k!}{l! (k-l)!} 2^l x^l t^{2k-l} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{(-1)^{k-l} (2k-1)!!}{2^{k-l} l! (k-l)!} x^l t^{2k-l} \end{aligned}$$

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$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{(-1)^{k-l} (2k-1)!!}{2^{k-l} l! (k-l)!} x^l t^{2k-l}$$

$$\begin{array}{l} l = n - 2m \\ n = 2k - l \end{array} \longrightarrow n = 2k - n + 2m \longrightarrow n = k + m$$

$$0 \leq l = n - 2m \leq k = n - m \longrightarrow 2m \leq n \leq n + m \longrightarrow$$

$$\begin{array}{l} 2m \leq n \\ 0 \leq m \end{array}$$

$$\begin{aligned} \frac{1}{\sqrt{1-2tx+t^2}} &= \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m-1)!!}{2^m m! (n-2m)!} x^{n-2m} t^n \\ &= \sum_{n=0}^{\infty} P_n(x) t^n \end{aligned}$$

ロドリグの公式

$$P_n(x) = \frac{1}{(2n)!!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$(x^2 - 1)^n = \sum_{k=0}^n \frac{n!}{k! (n-k)!} (-1)^k x^{2n-2k}$$

$$\frac{1}{(2n)!!} \frac{d^n}{dx^n} (x^2 - 1)^n = \frac{1}{(2n)!!} \frac{d^n}{dx^n} \sum_{k=0}^n \frac{n!}{k! (n-k)!} (-1)^k x^{2n-2k}$$

$$= \frac{1}{(2n)!!} \sum_{k=0}^{[n/2]} \frac{n! (2n-2k)!}{k! (n-k)! (2n-2k-n)!} (-1)^k x^{2n-2k-n}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^k k! 2^{n-k} (n-k)! (n-2k)!} x^{n-2k}$$

$$= \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{(2k)!! (2n-2k)!! (n-2k)!} x^{n-2k} = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k-1)!!}{(2k)!! (n-2k)!} x^{n-2k}$$

$$= P_n(x)$$

ルジャンドル関数の直交性

$$\int_{-1}^1 P_m(x) P_n(x) dx = 0 \quad (m \neq n)$$

$$\int_{-1}^1 x^m P_n(x) dx = \frac{1}{(2n)!!} \int_{-1}^1 x^m \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

$$= \frac{1}{(2n)!!} \left[x^m \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \right]_{-1}^1 - \frac{m}{(2n)!!} \int_{-1}^1 x^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx$$

$$= -\frac{m}{(2n)!!} \int_{-1}^1 x^{m-1} \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx$$

$$= \frac{(-1)^k m!}{(m-k)! (2n)!!} \int_{-1}^1 x^{m-k} \frac{d^{n-k}}{dx^{n-k}} (x^2 - 1)^n dx \quad (k = m, \quad m < n)$$

$$= \frac{(-1)^m m!}{(2n)!!} \int_{-1}^1 \frac{d^{n-m}}{dx^{n-m}} (x^2 - 1)^n dx = \frac{(-1)^m m!}{(2n)!!} \left[\frac{d^{n-m-1}}{dx^{n-m-1}} (x^2 - 1)^n \right]_{-1}^1 = 0$$

ルジャンドル関数の直交性

$(m > n)$

$$\int_{-1}^1 x^m P_n(x) dx = \frac{1}{(2n)!!} \int_{-1}^1 x^m \frac{d^n}{dx^n} (x^2 - 1)^n dx = \frac{(-1)^m m!}{(m-n)!(2n)!!} \int_{-1}^1 x^{m-n} (x^2 - 1)^n dx$$

$$\int_{-1}^1 x^{m-n} (x^2 - 1)^n dx = \left[\frac{x^{m-n+1}}{m-n-1} (x^2 - 1)^n \right]_{-1}^1 - \frac{2n}{m-n-1} \int_{-1}^1 x^{m-n+2} (x^2 - 1)^{n-1} dx$$

$$= (-1)^n \frac{2^n n!}{(m-n+1)(m-n+3) \cdots (m-n+2n-1)} \int_{-1}^1 x^{m-n+2} (x^2 - 1)^{n-1} dx$$

$$= \begin{cases} 0 & (m-n: \text{奇数}) \\ (-1)^n \frac{2(2n)!! (m-n-1)!!}{(m-n+2n+1)!!} & (m-n: \text{偶数}) \end{cases}$$

$$\int_{-1}^1 x^m P_n(x) dx = \begin{cases} 0 & (0 \leq m < n \text{ or } m-n: \text{奇数}) \\ \frac{2m!}{(m-n)!! (m+n+1)!!} & (n \leq m \text{ and } m-n: 0 \text{ or 偶数}) \end{cases}$$

$$(m-n)! = (m-n)!! (n-n-1)!!$$

ルジャンドル関数の直交性

$(m \geq n)$

$$\int_{-1}^1 x^m P_n(x) dx = \frac{1}{(2n)!!} \int_{-1}^1 x^m \frac{d^n}{dx^n} (x^2 - 1)^n dx = \frac{(-1)^m m!}{(m-n)!(2n)!!} \int_{-1}^1 x^{m-n} (x^2 - 1)^n dx$$

$$\int_{-1}^1 x^{m-n} (x^2 - 1)^n dx = \left[\frac{x^{m-n+1}}{m-n+1} (x^2 - 1)^n \right]_{-1}^1 - \frac{2n}{m-n+1} \int_{-1}^1 x^{m-n+2} (x^2 - 1)^{n-1} dx$$

$$= (-1)^n \frac{2^n n!}{(m-n+1)(m-n+3) \cdots (m-n+2n-1)} \int_{-1}^1 x^{m-n+2n} dx$$

$$= \begin{cases} 0 & (m-n: \text{奇数}) \\ (-1)^n \frac{2(2n)!! (m-n-1)!!}{(m-n+2n+1)!!} & (m-n: \text{偶数}) \end{cases}$$

$$\int_{-1}^1 x^m P_n(x) dx = \begin{cases} 0 & (0 \leq m < n \text{ or } m-n: \text{奇数}) \\ \frac{2m!}{(m-n)!! (m+n+1)!!} & (n \leq m \text{ and } m-n: 0 \text{ or 偶数}) \end{cases}$$

$$(m-n)! = (m-n)!! (n-n-1)!!$$

ルジャンドル関数の直交性

$$\int_{-1}^1 P_m(x)P_n(x)dx = 0 \quad (m \neq n)$$

$$\int_{-1}^1 P_n(x)P_n(x)dx = \frac{(2n-1)!!}{n!} \int_{-1}^1 x^n P_n(x)dx = \frac{(2n-1)!!}{n!} \frac{2n!}{(2n+1)!!} = \frac{2}{2n+1}$$

$$P_n(x) \text{ の最高次の係数 } a_n = \frac{(2n-1)!!}{n!}$$

$$\int_{-1}^1 x^m P_n(x)dx = \begin{cases} 0 & (0 \leq m < n \text{ or } m-n: \text{奇数}) \\ \frac{2m!}{(m-n)!!(m+n+1)!!} & (n \leq m \text{ and } m-n: 0 \text{ or 偶数}) \end{cases}$$

ルジャンドルの陪微分方程式

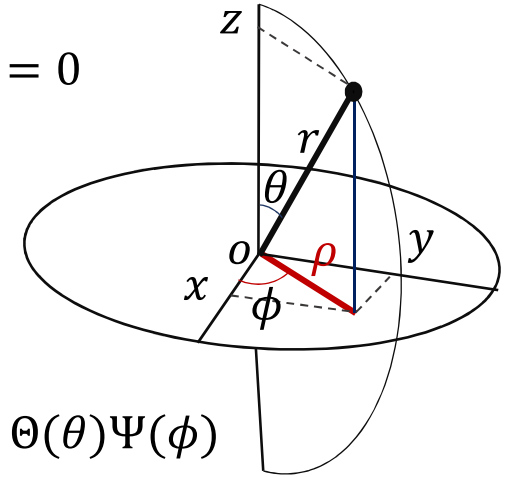
$Y(\theta, \phi)$ に ϕ 依存性がある場合 $\nabla^2 f = -\lambda f$ $f = R(r)Y(\theta, \phi)$

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y(\theta, \phi)}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y(\theta, \phi)}{\partial \phi^2} + n(n+1)Y(\theta, \phi) = 0$$

$$\frac{\sin \theta}{\Theta(\phi)} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Theta(\phi)}{\partial \theta} \right) + n(n+1) \sin^2 \theta = -\frac{1}{\Psi(\phi)} \frac{\partial^2 \Psi(\phi)}{\partial \phi^2}$$

$$\gamma \Psi(\phi) = -\frac{\partial^2 \Psi(\phi)}{\partial \phi^2} \quad \Psi(\phi) = e^{im\phi} \quad \gamma = m^2$$

$$Y(\theta, \phi) = \Theta(\theta)\Psi(\phi)$$



$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Theta(\phi)}{\partial \theta} \right) + \left(n(n+1) - \frac{m^2}{\sin^2 \theta} \right) \Theta(\phi) = 0$$

$$x = \cos \theta \quad dx = -\sin \theta d\theta = -\sqrt{1-x^2} d\theta$$

Legendre の陪微分方程式

$$\frac{d}{dx} \left((1-x^2) \frac{dP_n^m(x)}{dx} \right) + \left(n(n+1) - \frac{m^2}{1-x^2} \right) P_n^m(x) = 0$$