

ルジャンドルの陪微分方程式

$$\frac{d}{dx} \left((1-x^2) \frac{dP_n^m(x)}{dx} \right) + \left(n(n+1) - \frac{m^2}{1-x^2} \right) P_n^m(x) = 0 \quad P_n^m(x) = (1-x^2)^{m/2} y(x)$$

$$\begin{aligned} \frac{d}{dx} \left((1-x^2) \frac{dP_n^m(x)}{dx} \right) &= \frac{d}{dx} \left\{ (1-x^2) \left(-mx(1-x^2)^{m/2-1} y(x) + (1-x^2)^{m/2} \frac{dy(x)}{dx} \right) \right\} \\ &= \frac{d}{dx} \left(-mx(1-x^2)^{m/2} y(x) + (1-x^2)^{m/2+1} \frac{dy(x)}{dx} \right) \\ &= (1-x^2)^{m/2} \left\{ \left(-m + \frac{m^2 x^2}{1-x^2} \right) y(x) - 2(m+1)x \frac{dy(x)}{dx} + (1-x^2) \frac{d^2 y(x)}{dx^2} \right\} \end{aligned}$$

$$-m + \frac{m^2 x^2}{1-x^2} = -m + \frac{m^2}{1-x^2} - \frac{m^2}{1-x^2} + \frac{m^2 x^2}{1-x^2} = -m + \frac{m^2}{1-x^2} - \frac{m^2(1-x^2)}{1-x^2} = -m(m+1) + \frac{m^2}{1-x^2}$$

Legendre の陪微分方程式

$$(1-x^2) \frac{d^2 y(x)}{dx^2} - 2(m+1)x \frac{dy(x)}{dx} + \{n(n+1) - m(m+1)\} y(x) = 0$$

ルジャンドルの微分方程式の変形

テキスト:p. 55

Legendre の微分方程式

$$\frac{d}{dx} \left((1-x^2) \frac{dP_n(x)}{dx} \right) + n(n+1)P_n(x) = 0$$

$$\frac{d^m}{dx^m} \{xf(x)\} = \frac{d^{m-1}}{dx^{m-1}} \left\{ f(x) + x \frac{d}{dx} f(x) \right\}$$

Legendre の微分方程式のm階微分

$$\frac{d^m}{dx^m} \left\{ (1-x^2) \frac{d^2 P_n(x)}{dx^2} - 2x \frac{dP_n(x)}{dx} + n(n+1)P_n(x) \right\} = 0$$

$$= m \frac{d^{m-1}}{dx^{m-1}} f(x) + x \frac{d^m}{dx^m} f(x)$$

$$\frac{d^m}{dx^m} \{x^2 f(x)\} = \frac{d^{m-1}}{dx^{m-1}} \left\{ 2xf(x) + x^2 \frac{d}{dx} f(x) \right\}$$

$$= \frac{d^{m-1}}{dx^{m-1}} \{2xf(x)\} + \frac{d^{m-2}}{dx^{m-2}} \left\{ 2x \frac{d}{dx} f(x) + x^2 \frac{d^2}{dx^2} f(x) \right\}$$

$$= \frac{d^{m-1}}{dx^{m-1}} \{2xf(x)\} + \frac{d^{m-2}}{dx^{m-2}} \left\{ 2x \frac{d}{dx} f(x) \right\} + \frac{d^{m-3}}{dx^{m-3}} \left\{ 2x \frac{d^2}{dx^2} f(x) + x^2 \frac{d^3}{dx^3} f(x) \right\}$$

$$= \sum_{l=1}^m \frac{d^{m-l}}{dx^{m-l}} \left\{ 2x \frac{d^{l-1}}{dx^{l-1}} f(x) \right\} + x^2 \frac{d^m}{dx^m} f(x)$$

$$= \sum_{l=1}^m \left\{ 2(m-l) \frac{d^{m-2}}{dx^{m-2}} f(x) + 2x \frac{d^{m-1}}{dx^{m-1}} f(x) \right\} + x^2 \frac{d^m}{dx^m} f(x)$$

ルジャンドル陪関数

$$\frac{d^m}{dx^m} \{xf(x)\} = m \frac{d^{m-1}}{dx^{m-1}} f(x) + x \frac{d^m}{dx^m} f(x)$$

$$\frac{d^m}{dx^m} \{x^2 f(x)\} = m(m-1) \frac{d^{m-2}}{dx^{m-2}} f(x) + 2mx \frac{d^{m-1}}{dx^{m-1}} f(x) + x^2 \frac{d^m}{dx^m} f(x)$$

Legendre の微分方程式のm階微分

$$\frac{d^m}{dx^m} \left\{ (1-x^2) \frac{d^2 P_n(x)}{dx^2} - 2x \frac{dP_n(x)}{dx} + n(n+1)P_n(x) \right\} = 0$$

$$(1-x^2) \frac{d^{m+2}}{dx^{m+2}} P_n(x) - (2m+2)x \frac{d^{m+1}}{dx^{m+1}} P_n(x) + \{n(n+1) - 2m - m(m-1)\} \frac{d^m}{dx^m} P_n(x) = 0$$

$$(1-x^2) \frac{d^2}{dx^2} \frac{d^m P_n(x)}{dx^m} - 2(m+1)x \frac{d}{dx} \frac{d^m P_n(x)}{dx^m} + \{n(n+1) - m(m+1)\} \frac{d^m P_n(x)}{dx^m} = 0$$

Legendre の陪微分方程式

$$(1-x^2) \frac{d^2 y(x)}{dx^2} - 2(m+1)x \frac{dy(x)}{dx} + \{n(n+1) - m(m+1)\} y(x) = 0$$

Legendre 陪関数 $P_n^m(x) = (1-x^2)^{\frac{m}{2}} y(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}$

ルジャンドル陪関数

ロドリゲの公式

テキスト: p. 56

$$P_n(x) = \frac{1}{(2n)!!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} = \frac{1}{(2n)!!} (1 - x^2)^{m/2} \frac{d^{m+n}}{dx^{m+n}} (x^2 - 1)^n$$

Legendre 陪関数

Legendre 陪関数の直交性

$$\begin{aligned} \int_{-1}^1 P_l^m(x) P_n^m(x) dx &= \int_{-1}^1 (1 - x^2)^m \frac{d^m P_l(x)}{dx^m} \frac{d^m P_n(x)}{dx^m} dx \\ &= \left[\frac{d^{m-1} P_l(x)}{dx^{m-1}} \frac{d^m P_n(x)}{dx^m} (1 - x^2)^m \right]_{-1}^1 \\ &\quad - \int_{-1}^1 \frac{d^{m-1} P_l(x)}{dx^{m-1}} \frac{d}{dx} \left\{ \frac{d^m P_n(x)}{dx^m} (1 - x^2)^m \right\} dx \\ &= - \int_{-1}^1 \frac{d^{m-1} P_l(x)}{dx^{m-1}} \frac{d}{dx} \left\{ (1 - x^2)^m \frac{d^m P_n(x)}{dx^m} \right\} dx \end{aligned}$$

ルジャンドル陪関数の直交関係

$$\int_{-1}^1 (1-x^2)^m \frac{d^m P_l(x)}{dx^m} \frac{d^m P_n(x)}{dx^m} dx = - \int_{-1}^1 \frac{d^{m-1} P_l(x)}{dx^{m-1}} \frac{d}{dx} \left\{ (1-x^2)^m \frac{d^m P_n(x)}{dx^m} \right\} dx$$

(621)式において $m \rightarrow m-1$ に修正し $(1-x^2)^{m-1}$ をかける

$$\begin{aligned} (1-x^2)^m \frac{d}{dx} \frac{d^m P_n(x)}{dx^m} - 2mx(1-x^2)^{m-1} \frac{d^m P_n(x)}{dx^m} \\ + \{n(n+1) - m(m-1)\} (1-x^2)^{m-1} \frac{d^{m-1} P_n(x)}{dx^{m-1}} = 0 \end{aligned}$$

$$\frac{d}{dx} \left\{ (1-x^2)^m \frac{d^m P_n(x)}{dx^m} \right\} = - \underbrace{\{n(n+1) - m(m-1)\}}_{(n-m+1)(n+m)} (1-x^2)^{m-1} \frac{d^{m-1} P_n(x)}{dx^{m-1}}$$

$$\begin{aligned} \int_{-1}^1 (1-x^2)^m \frac{d^m P_l(x)}{dx^m} \frac{d^m P_n(x)}{dx^m} dx \\ = (n-m+1)(n+m) \int_{-1}^1 (1-x^2)^{m-1} \frac{d^{m-1} P_l(x)}{dx^{m-1}} \frac{d^{m-1} P_n(x)}{dx^{m-1}} dx \end{aligned}$$

ルジャンドル陪関数の直交関係

$$\begin{aligned}
\int_{-1}^1 P_l^m(x) P_n^m(x) dx &= \int_{-1}^1 (1-x^2)^m \frac{d^m P_l(x)}{dx^m} \frac{d^m P_n(x)}{dx^m} dx \\
&= (n-m+1)(n+m) \int_{-1}^1 (1-x^2)^{m-1} \frac{d^{m-1} P_l(x)}{dx^{m-1}} \frac{d^{m-1} P_n(x)}{dx^{m-1}} dx \\
&= (n-m+1)(n-m+2)(n+m)(n+m-1) \int_{-1}^1 (1-x^2)^{m-2} \frac{d^{m-2} P_l(x)}{dx^{m-2}} \frac{d^{m-2} P_n(x)}{dx^{m-2}} dx \\
&= (n-m+1)(n-m+2) \cdots n(n+m)(n+m-1) \cdots (n+1) \int_{-1}^1 P_l(x) P_n(x) dx \\
&= \frac{(n+m)!}{(n-m)!} \int_{-1}^1 P_l(x) P_n(x) dx = \frac{(n+m)!}{(n-m)!} \frac{2}{2n+1} \delta_{ln}
\end{aligned}$$

球面調和関数

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y(\theta, \phi)}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y(\theta, \phi)}{\partial \phi^2} + l(l+1)Y(\theta, \phi) = 0$$

$$Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$$

$$\Phi(\phi) = e^{im\phi}$$

$$\Theta(\theta) = P_l^m(x)$$

球面調和関数

$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im}$$

$l = 0$

s軌道

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$$

$l = 1$

p軌道

$$Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$l = 2$

d軌道

$$Y_2^0(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$$

$$Y_2^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_2^{\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i}$$

$$P_n^m(x) = \frac{1}{(2n)!!} (1-x^2)^{m/2} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n$$

球面調和関数

