

調和振動子の量子化

テキスト: p. 59

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} u(x) + \frac{1}{2} K x^2 u(x) = E u(x)$$

$$-\frac{d^2 u}{d\xi^2} + (\xi^2 - \lambda) u = 0$$

$$\begin{aligned} \alpha^2 &= \sqrt{mK}/\hbar & \xi &= \alpha x \\ \omega_c &= \sqrt{K/m} & \lambda &= \frac{2E}{\hbar\omega_c} \end{aligned}$$

$$\frac{d^2 u}{d\xi^2} = \xi^2 u \quad (\xi \rightarrow \infty) \quad \Rightarrow \quad u(\xi) = H(\xi) e^{-\frac{1}{2}\xi^2}$$

$$\begin{aligned} \frac{d^2 u}{d\xi^2} &= \frac{d}{d\xi} \left(\frac{dH(\xi)}{d\xi} e^{-\frac{1}{2}\xi^2} - \xi H(\xi) e^{-\frac{1}{2}\xi^2} \right) \\ &= \frac{d^2 H(\xi)}{d\xi^2} e^{-\frac{1}{2}\xi^2} - 2\xi \frac{dH(\xi)}{d\xi} e^{-\frac{1}{2}\xi^2} - H(\xi) e^{-\frac{1}{2}\xi^2} + \xi^2 H(\xi) e^{-\frac{1}{2}\xi^2} \end{aligned}$$

$$\frac{d^2 H(\xi)}{d\xi^2} e^{-\frac{1}{2}\xi^2} - 2\xi \frac{dH(\xi)}{d\xi} e^{-\frac{1}{2}\xi^2} + (\lambda - 1) H(\xi) e^{-\frac{1}{2}\xi^2} = 0$$

エルミート多項式

テキスト: p. 60

$$\frac{d^2 H(x)}{dx^2} - 2x \frac{dH(x)}{dx} + (\lambda - 1)H(x) = 0$$

$$H(x) = \sum_{m=0}^{\infty} a_m x^{s+m} = x^s (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$$

$$\frac{d^2 H(x)}{dx^2} = \sum_{m=0}^{\infty} (s+m)(s+m-1)a_m x^{s+m-2} \quad \frac{dH(x)}{dx} = \sum_{m=0}^{\infty} (s+m)a_m x^{s+m-1}$$

$$\sum_{m=0}^{\infty} (s+m)(s+m-1)a_m x^{s+m-2} - 2 \sum_{m=0}^{\infty} (s+m)a_m x^{s+m-1} + (\lambda - 1) \sum_{m=0}^{\infty} a_m x^{s+m} = 0$$

$$x^{s-2} \text{ の係数 } s(s-1)a_0 = 0 \quad x^{s-1} \text{ の係数 } (s+1)a_1 = 0$$

$$(s+m+2)(s+m+1)a_{m+2} - 2(s+m)a_m + (\lambda - 1)a_m = 0$$

エルミート多項式

テキスト: p. 60

$$(s + m + 2)(s + m + 1)a_{m+2} - 2(s + m)a_m + (\lambda - 1)a_m = 0$$

$$a_{m+2} = \frac{2(s + m) + 1 - \lambda}{(s + m + 2)(s + m + 1)} a_m$$

$$H(x) = \sum_{l=0} c_{2l+1} x^{2l+1}$$

$$c_{2(l+1)+1} = \frac{2(2l + 1) + 1 - \lambda}{(2l + 3)(2l + 2)} c_{2l+1}$$

$$H(x) = \sum_{l=0} c_{2l} x^{2l}$$

$$c_{2(l+1)} = \frac{2(2l) + 1 - \lambda}{(2l + 2)(2l + 1)} c_{2l}$$

級数の漸近形は e^{x^2} と同じ →

発散しないための切断条件が必要

$$n : \text{最大次数} \quad \lambda = 2n + 1$$

エルミートの微分方程式

$$\frac{d^2 H(x)}{dx^2} - 2x \frac{dH(x)}{dx} + 2nH(x) = 0$$

固有エネルギーの離散化

$$E = \frac{\hbar \omega_c}{2} \lambda$$

エルミート多項式

テキスト: p. 61

$$\frac{d^2 H(x)}{dx^2} - 2x \frac{dH(x)}{dx} + 2nH(x) = 0$$

$$c_n = 2^n$$

$$c_{n-2} = -\frac{n(n-1)}{2(n-n+2)} 2^n$$

$$c_m = -\frac{2(n-m+2)}{m(m-1)} c_{m-2}$$

$$c_{n-4} = (-1)^2 \frac{n(n-1)(n-2)(n-3)}{2(2)2(4)} 2^n$$

$$c_{n-6} = (-1)^3 \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2(2)2(4)2(6)} 2^n$$

$$c_{n-2l} = (-1)^l \frac{n!}{(n-2l)! 2^l (2l)!!} 2^n$$

$$H_n(x) = \sum_{l=0}^{[n/2]} (-1)^l \frac{n!}{(n-2l)! l!} (2x)^{n-2l}$$

エルミート多項式

テキスト:p. 62

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_3(x) = 8x^3 - 12x$$

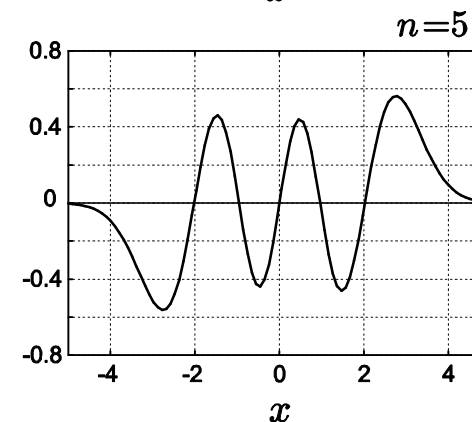
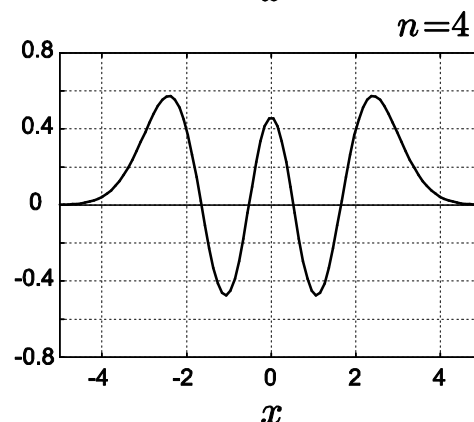
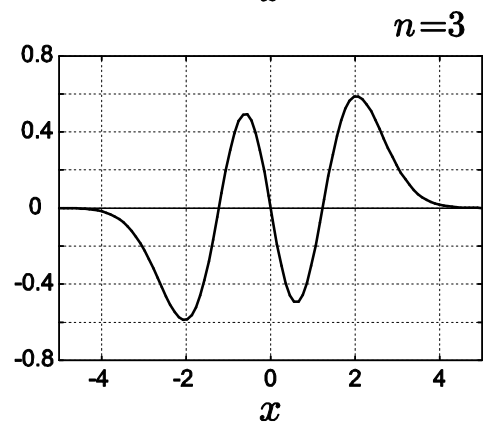
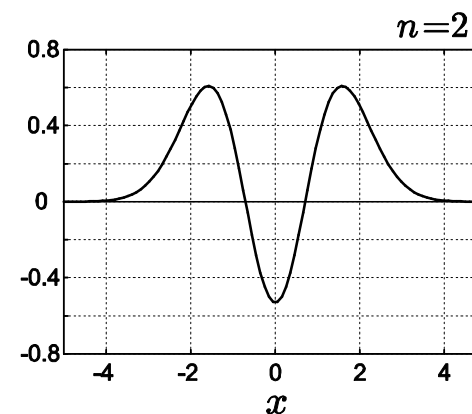
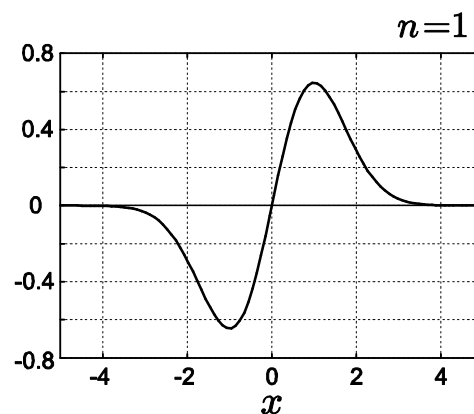
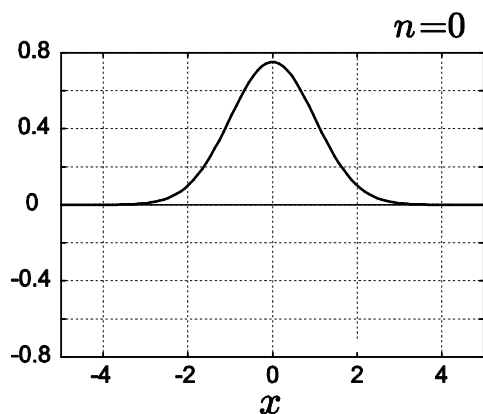
$$H_2(x) = 4x^2 - 2 \quad H_4(x) = 16x^4 - 48x^2 + 12$$

$$c_n = 2^n$$

$$c_{n-2} = -\frac{n(n-1)}{2(2)} 2^n$$

$$c_{n-4} = \frac{n(n-1)(n-2)(n-3)}{2(2)2(4)} 2^n$$

$$H_n(x) e^{-\frac{1}{2}x^2} (\sqrt{\pi} 2^n n!)^{-1/2}$$



エルミート多項式の母関数

$$\begin{aligned}
 e^{2xt-t^2} &= e^{2xt} e^{-t^2} \\
 &= \left(\sum_{m=0}^{\infty} \frac{2^m x^m t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \frac{(-1)^l t^{2l}}{l!} \right) \\
 &= \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^l 2^m x^m t^{m+2l}}{m! l!} \\
 &= \sum_{n=0}^{\infty} \sum_{l=0}^{[n/2]} \frac{(-1)^l (2x)^{n-2l} t^n}{(n-2l)! l!} \\
 &= \sum_{n=0}^{\infty} \sum_{l=0}^{[n/2]} \frac{(-1)^l n!}{(n-2l)! l!} (2x)^{n-2l} \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \sum_{l=0}^{[n/2]} H_n(x) \frac{t^n}{n!}
 \end{aligned}$$

$$m = n - 2l$$

$$0 < m \longrightarrow 2l < n$$

エルミート多項式の母関数

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

エルミート多項式の直交性

$$e^{2xt-t^2}e^{2xs-s^2} = \left(\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \right) \left(\sum_{m=0}^{\infty} \frac{H_m(x)}{m!} s^m \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_n(x) H_m(x) \frac{t^n s^m}{n! m!}$$

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2xt-t^2} e^{2xs-s^2} e^{-x^2} dx &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n s^m}{n! m!} \int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx \\ &= \int_{-\infty}^{\infty} e^{-(x-t-s)^2+2} dx = e^{2st} \int_{-\infty}^{\infty} e^{-(x-t-s)^2} dx \\ &= e^{2st} \sqrt{\pi} = \sqrt{\pi} \sum_{m=0}^{\infty} \frac{2^n t^n s^n}{n!} \end{aligned}$$

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{nm}$$

エルミート多項式の漸化式

$$\frac{\partial}{\partial x} e^{2xt-t^2} = 2te^{2xt-t^2} = 2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

$$\frac{\partial}{\partial x} e^{2xt-t^2} = \frac{\partial}{\partial x} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{\partial H_n(x)}{\partial x} \frac{1}{n!} t^n$$

$$2nH_{n-1}(x) = \frac{\partial H_n(x)}{\partial x}$$



$$2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{\partial H_n(x)}{\partial x} \frac{1}{n!} t^n \Rightarrow 2 \frac{H_{n-1}(x)}{(n-1)!} = \frac{\partial H_n(x)}{\partial x} \frac{1}{n!}$$

$$\frac{\partial}{\partial t} e^{2xt-t^2} = (2x + 2t)e^{2xt-t^2} = (2x - 2t) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

$$\frac{\partial}{\partial t} e^{2xt-t^2} = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = \sum_{n=0}^{\infty} nH_n(x) \frac{1}{n!} t^{n-1}$$

$$2x \frac{H_n(x)}{n!} - 2 \frac{H_{n-1}(x)}{(n-1)!} = (n+1) \frac{H_{n+1}(x)}{(n+1)!}$$

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)$$